

On some conjectures concerning critical independent sets of a graph

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Abstract

Let G be a simple graph with vertex set $V(G)$. A set $S \subseteq V(G)$ is independent if no two vertices from S are adjacent. For $X \subseteq V(G)$, the difference of X is $d(X) = |X| - |N(X)|$ and an independent set A is critical if $d(A) = \max\{d(X) : X \subseteq V(G) \text{ is an independent set}\}$ (possibly $A = \emptyset$). Let $\text{nucleus}(G)$ and $\text{diadem}(G)$ be the intersection and union, respectively, of all maximum size critical independent sets in G . In this paper, we will give two new characterizations of König-Egerváry graphs involving $\text{nucleus}(G)$ and $\text{diadem}(G)$. We also prove a related lower bound for the independence number of a graph. This work answers several conjectures posed by Jarden, Levit, and Mandrescu.

Keywords: maximum independent set, maximum critical independent set, König-Egerváry graph, maximum matching, core, corona, ker, diadem, nucleus.

1 Introduction

In this paper G is a simple graph with vertex set $V(G)$, $|V(G)| = n$, and edge set $E(G)$. The set of neighbors of a vertex v is $N_G(v)$ or simply $N(v)$ if there is no possibility of ambiguity. If $X \subseteq V(G)$, then the set of neighbors of X is $N(X) = \cup_{u \in X} N(u)$, $G[X]$ is the subgraph induced by X , and X^c is the complement of the subset X . For sets $A, B \subseteq V(G)$, we use $A \setminus B$ to denote the vertices belonging to A but not B . For such disjoint A and B we let (A, B) denote the set of edges such that each edge is incident to both a vertex in A and a vertex in B .

*Supported in part by the NSF DMS under contract 1300547.

A *matching* M is a set of pairwise non-incident edges of G . A matching of maximum cardinality is a *maximum matching* and $\mu(G)$ is the cardinality of such a maximum matching. For a set $A \subseteq V(G)$ and matching M , we say A is *saturated* by M if every vertex of A is incident to an edge in M . For two disjoint sets $A, B \subseteq V(G)$, we say there is a matching M of A into B if M is a matching of G such that every edge of M belongs to (A, B) and each vertex of A is saturated. An *M -alternating path* is a path that alternates between edges in M and those not in M . An *M -augmenting path* is an M -alternating path which begins and ends with an edge not in M .

A set $S \subseteq V(G)$ is *independent* if no two vertices from S are adjacent. An independent set of maximum cardinality is a *maximum independent set* and $\alpha(G)$ is the cardinality of such a maximum independent set. For a graph G , let $\Omega(G)$ denote the family of all its maximum independent sets, let

$$\text{core}(G) = \bigcap \{S : S \in \Omega(G)\}, \quad \text{and} \quad \text{corona}(G) = \bigcup \{S : S \in \Omega(G)\}.$$

See [1, 9, 14] for background and properties of $\text{core}(G)$ and $\text{corona}(G)$.

For a graph G and a set $X \subseteq V(G)$, the *difference* of X is $d(X) = |X| - |N(X)|$ and the *critical difference* $d(G)$ is $\max\{d(X) : X \subseteq V(G)\}$. Zhang [16] showed that $\max\{d(X) : X \subseteq V(G)\} = \max\{d(S) : S \subseteq V(G) \text{ is an independent set}\}$. The set X is a *critical set* if $d(X) = d(G)$. The set $S \subseteq V(G)$ a *critical independent set* if S is both a critical set and independent. A critical independent set of maximum cardinality is called a *maximum critical independent set*. Note that for some graphs the empty set is the only critical independent set, for example odd cycles or complete graphs. See [2, 7, 8, 16] for more background and properties of critical independent sets.

Finding a maximum independent set is a well-known **NP**-hard problem. Zhang [16] first showed that a critical independent set can be found in polynomial time. Butenko and Trukhanov [2] showed that every critical independent set is contained in a maximum independent set, thereby directly connecting the problem of finding a critical independent set to that of finding a maximum independent set.

For a graph G the inequality $\alpha(G) + \mu(G) \leq n$ always holds. A graph G is a *König-Egerváry graph* if $\alpha(G) + \mu(G) = n$. All bipartite graphs are König-Egerváry but there are non-bipartite graphs which are König-Egerváry as well, see Figure 2 for an example. We adopt the convention that the empty graph K_0 , without vertices, is a König-Egerváry graph. In [7] it was shown that König-Egerváry graphs are closely related to critical independent sets.

Theorem 1.1. [7] *A graph G is König-Egerváry if, and only if, every maximum independent set in G is critical.*

Theorem 1.2. [7] *For any graph G , there is a unique set $X \subseteq V(G)$ such that all of the following hold:*

- (i) $\alpha(G) = \alpha(G[X]) + \alpha(G[X^c])$,
- (ii) $G[X]$ is a König-Egerváry graph,
- (iii) for every non-empty independent set S in $G[X^c]$, $|N(S)| \geq |S|$, and
- (iv) for every maximum critical independent set I of G , $X = I \cup N(I)$.

Larson in [8] showed that a maximum critical independent set can be found in polynomial time. So the decomposition in Theorem 1.2 of a graph G into X and X^c is also computable in polynomial time. Figure 1 gives an example of this decomposition, where both the sets X and X^c are non-empty. Recall, for some graphs the empty set is the only critical independent set, so for such graphs the set X would be empty. If a graph G is a König-Egerváry graph, then the set X^c would be empty. We adopt the convention that if K_0 is empty graph, then $\alpha(K_0) = 0$.

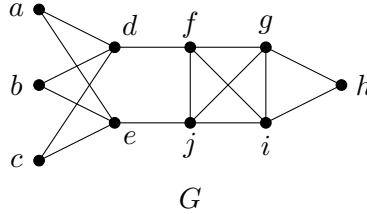


Figure 1: G has maximum critical independent set $I = \{a, b, c\}$. Theorem 1.2 gives that $X = \{a, b, c, d, e\}$ and $X^c = \{f, g, h, i, j\}$.

In [5, 11] the following concepts were introduced: for a graph G ,

$$\begin{aligned} \ker(G) &= \bigcap \{S : S \text{ is a critical independent set in } G\}, \\ \text{diadem}(G) &= \bigcup \{S : S \text{ is a critical independent set in } G\}, \text{ and} \\ \text{nucleus}(G) &= \bigcap \{S : S \text{ is a maximum critical independent set in } G\}. \end{aligned}$$

However, the following result due to Larson allows us to use a more suitable definition for $\text{diadem}(G)$.

Theorem 1.3. [8] *Each critical independent set is contained in some maximum critical independent set.*

For the remainder of this paper we define

$$\text{diadem}(G) = \bigcup \{S : S \text{ is a maximum critical independent set in } G\}.$$

Note that if G is a graph where the empty set is the only critical independent set (including the case $G = K_0$, the empty graph), then $\ker(G)$, $\text{diadem}(G)$, and $\text{nucleus}(G)$ are all empty. See Figure 2 for examples of the sets $\ker(G)$, $\text{diadem}(G)$, and $\text{nucleus}(G)$.

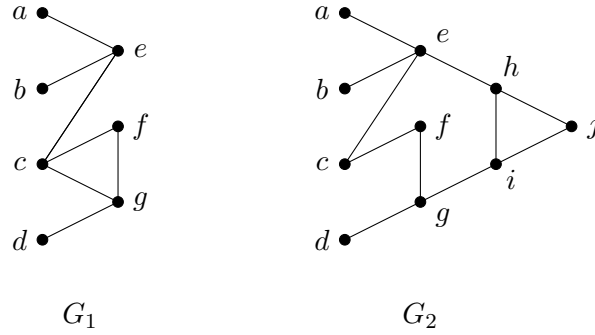


Figure 2: G_1 is a König-Egerváry graph with $\ker(G_1) = \{a, b\} \subsetneq \text{core}(G_1) = \text{nucleus}(G_1) = \{a, b, d\}$ and $\text{diadem}(G_1) = \text{corona}(G_1) = \{a, b, c, d, f\}$. G_2 is not a König-Egerváry graph and has $\ker(G_2) = \text{core}(G_2) = \{a, b\} \subsetneq \text{nucleus}(G_2) = \{a, b, d\}$ and $\text{diadem}(G_2) = \{a, b, c, d, f\} \subsetneq \text{corona}(G) = \{a, b, c, d, f, g, h, i, j\}$.

In [4, 5], the following necessary conditions for König-Egerváry graphs were given:

Theorem 1.4. [4] *If G is a König-Egerváry graph, then*

- (i) $\text{diadem}(G) = \text{corona}(G)$, and
- (ii) $|\ker(G)| + |\text{diadem}(G)| \leq 2\alpha(G)$.

Theorem 1.5. [5] *If G is a König-Egerváry graph, then $|\text{nucleus}(G)| + |\text{diadem}(G)| = 2\alpha(G)$.*

In [4] it was conjectured that condition (i) of Theorem 1.4 is sufficient for König-Egerváry graphs and in [5] it was conjectured the necessary condition in Theorem 1.5 is also sufficient. The purpose of this paper is to affirm these conjectures by proving the following new characterizations of König-Egerváry graphs.

Theorem 1.6. *For a graph G , the following are equivalent:*

- (i) G is a König-Egerváry graph,
- (ii) $\text{diadem}(G) = \text{corona}(G)$, and
- (iii) $|\text{diadem}(G)| + |\text{nucleus}(G)| = 2\alpha(G)$.

The paper [4] gives an upper bound for $\alpha(G)$ in terms of unions and intersections of maximum independent sets, proving

$$2\alpha(G) \leq |\text{core}(G)| + |\text{corona}(G)|$$

for any graph G . It is natural to ask whether a similar lower bound for $\alpha(G)$ can be formulated in terms of unions and intersections of critical independent sets. Jarden, Levit, and Mandrescu in [4] conjectured that for any graph G , the inequality $|\ker(G)| + |\text{diadem}(G)| \leq 2\alpha(G)$ always holds. We will prove a slightly stronger statement. By Theorem 1.3 we see that $\ker(G) \subseteq \text{nucleus}(G)$ holds implying that $|\ker(G)| + |\text{diadem}(G)| \leq |\text{nucleus}(G)| + |\text{diadem}(G)|$. In section 4 we will prove the following statement, resolving the cited conjecture:

Theorem 1.7. *For any graph G ,*

$$|\text{nucleus}(G)| + |\text{diadem}(G)| \leq 2\alpha(G).$$

It would be interesting to know whether the sets $\text{nucleus}(G)$ and $\text{diadem}(G)$, or their sizes, can be computed in polynomial time.

2 Some structural lemmas

Here we prove several crucial lemmas which will be needed in our proofs. Our results hinge upon the structure of the set X as described in Theorem 1.2.

Lemma 2.1. *Let I be a maximum critical independent set in G and set $X = I \cup N(I)$. Then $\text{diadem}(G) \cup N(\text{diadem}(G)) = X$.*

Proof. By Theorem 1.2 the set X is unique in G , that is, for any maximum critical independent set S , $X = S \cup N(S)$. Then $\text{diadem}(G) = X$ follows by definition. \square

Lemma 2.2. *Let I be a maximum critical independent set in G and set $X = I \cup N(I)$. Then $\text{diadem}(G) \subseteq \text{diadem}(G[X])$ and $\text{nucleus}(G[X]) \subseteq \text{nucleus}(G)$.*

Proof. Let S be a maximum critical independent set in G . Using Theorem 1.2 we see that S is a maximum independent set in $G[X]$ and also $G[X]$ is a König-Egerváry graph. Then Theorem 1.1 gives that S must also be critical in $G[X]$, which implies that $\text{diadem}(G) \subseteq \text{diadem}(G[X])$.

Now let $v \in \text{nucleus}(G[X])$. Then v belongs to every maximum critical independent set in $G[X]$. As remarked above, since every maximum critical independent set in G is also a maximum critical independent set in $G[X]$, then v belongs to every maximum critical independent set in G . This shows that $v \in \text{nucleus}(G)$ and $\text{nucleus}(G[X]) \subseteq \text{nucleus}(G)$ follows. \square

Lemma 2.3. *Suppose I is a non-empty maximum critical independent set in G , set $X = I \cup N(I)$, let $A = \text{nucleus}(G) \setminus \text{nucleus}(G[X])$, and let S be a maximum independent set in $G[X]$. For $S' \subseteq S \cap N(A)$, if there exists $A' \subseteq A$ such that $N(A') \cap S \subseteq S'$, then $|S'| \geq |A'|$.*

Proof. For $S' \subseteq S \cap N(A)$ suppose such an A' exists. For sake of contradiction, suppose that $|S'| < |A'|$. Since $A' \subseteq \text{nucleus}(G)$, then A' is an independent set. Also since $A' \subseteq \text{nucleus}(G) \subseteq \text{diadem}(G)$, by Lemma 2.1 we have $A' \subseteq X$. Furthermore, since $N(A') \cap S \subseteq S'$ then $A' \cup (S \setminus S')$ is an independent set in $G[X]$. Now by assumption $|S'| < |A'|$, so $A' \cup (S \setminus S')$ is an independent set in $G[X]$ larger than S , which cannot happen. Therefore we must have $|S'| \geq |A'|$ as desired. \square

Lemma 2.4. *Let I be a maximum critical independent set in G and set $X = I \cup N(I)$. Then*

$$|\text{nucleus}(G)| + |\text{diadem}(G)| \leq |\text{nucleus}(G[X])| + |\text{diadem}(G[X])|.$$

Proof. First note that if the set X is empty, then by Lemma 2.1 both sides of the inequality are zero. So let us assume that X is non-empty. Now consider the set $A = \text{nucleus}(G) \setminus \text{nucleus}(G[X])$. If this independent set is empty, then $\text{nucleus}(G) = \text{nucleus}(G[X])$ and there is nothing to prove since $\text{diadem}(G) \subseteq \text{diadem}(G[X])$ holds by Lemma 2.2. If A is non-empty, for each $v \in A$ there is some maximum independent set S of $G[X]$ which doesn't contain v . Since S is a maximum independent set there exists $u \in N(v) \cap S$. Since $v \in \text{nucleus}(G)$, then u does not belong to any maximum critical independent set in G . Recall by Theorem 1.2 (ii) $G[X]$ is a König-Egerváry graph, so Theorem 1.1 gives that S is a maximum critical independent set in $G[X]$. It follows that $u \in \text{diadem}(G[X]) \setminus \text{diadem}(G)$, which shows each vertex in A is adjacent to at least one vertex in $\text{diadem}(G[X]) \setminus \text{diadem}(G)$.

Now we will show there is a maximum matching from A into $\text{diadem}(G[X]) \setminus \text{diadem}(G)$ with size $|A|$. For sake of contradiction, suppose such a matching M has less than $|A|$ edges. Then there exists some vertex $v \in A$ not saturated by M . By the above, v is adjacent to some vertex $u \in \text{diadem}(G[X]) \setminus \text{diadem}(G)$. Since M is maximum, u is matched to some vertex $w \in A$ under M . Now let S be a maximum independent set of $G[X]$ containing u . We now restrict ourselves to the subgraph induced by the edges $(A \cap N(S), S \cap N(A))$, noting this subgraph is bipartite since both $A \cap N(S)$ and $S \cap N(A)$ are independent. In this subgraph, consider the set \mathcal{P} of all M -alternating paths starting with the edge vu . Note that all such paths must start with the vertices v, u , then w . Also, such paths must end at either a matched vertex in $A \cap N(S)$ or an unmatched vertex in $S \cap N(A)$.

We wish to show that there is some alternating path ending at an unmatched vertex in $S \cap N(A)$. For sake of contradiction, suppose all alternating paths end at a matched vertex in $A \cap N(S)$ and let $V(\mathcal{P})$ denote the union of all vertices belonging to such an alternating path. We aim to show this scenario contradicts Lemma 2.3. Now clearly we must have $N(V(\mathcal{P}) \cap A) \cap S \subseteq V(\mathcal{P}) \cap S$, else we could extend an alternating path to any vertex in $(N(V(\mathcal{P}) \cap A) \cap S) \setminus (V(\mathcal{P}) \cap S)$. Also, since all paths in \mathcal{P} end at a matched vertex in $A \cap N(S)$, then every vertex of $V(\mathcal{P}) \cap S$ is matched under M , and such a situation should look as in Figure 3.

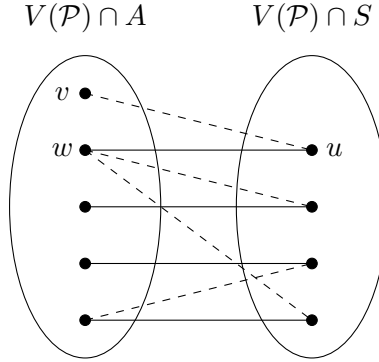


Figure 3: What the M -alternating paths could look like between $V(\mathcal{P}) \cap A$ and $V(\mathcal{P}) \cap S$, where solid lines represent matched edges in M and dotted lines represent the unmatched edges.

From this it follows that $|V(\mathcal{P}) \cap S| < |V(\mathcal{P}) \cap A|$. The previous statements exactly contradict Lemma 2.3, so there is some alternating path P ending at

an unmatched vertex $x \in S \cap N(A)$. This means that P is an M -augmenting path. A well-known theorem in graph theory states that a matching is maximum in G if, and only if, there is no augmenting path [15]. So P being an M -augmenting path contradicts our assumption that M is a maximum matching.

Therefore there is a matching M from A into $\text{diadem}(G[X]) \setminus \text{diadem}(G)$. This matching implies that $|\text{nucleus}(G) \setminus \text{nucleus}(G[X])| \leq |\text{diadem}(G[X]) \setminus \text{diadem}(G)|$. Since both $\text{nucleus}(G[X]) \subseteq \text{nucleus}(G)$ and $\text{diadem}(G) \subseteq \text{diadem}(G[X])$ by Lemma 2.2, the lemma follows. \square

3 New characterizations of König-Egerváry graphs

Proof (of Theorem 1.6). First we prove $(ii) \Rightarrow (i)$. Suppose that $\text{diadem}(G) = \text{corona}(G)$ holds and let I be a maximum critical independent set with $X = I \cup N(I)$. We will use the decomposition in Theorem 1.2 to show that X^c must be empty and hence, $G = G[X]$ is a König-Egerváry graph. By Lemma 2.1 we have $\text{corona}(G) = \text{diadem}(G) \subseteq X$, in other words every maximum independent set in G is contained in X . This implies that $|I| = \alpha(G[X]) = \alpha(G)$. Now by Theorem 1.2 (i), $\alpha(G) = \alpha(G[X]) + \alpha(G[X^c])$ showing that we must have $\alpha(G[X^c]) = 0$. Now clearly the result follows, since $\alpha(G[X^c]) = 0$ implies that X^c must be empty.

To prove $(iii) \Rightarrow (i)$, again we will use the decomposition in Theorem 1.2 to show that X^c must be empty and hence, G is a König-Egerváry graph. So suppose that $|\text{diadem}(G)| + |\text{nucleus}(G)| = 2\alpha(G)$ and let I be a maximum critical independent set in G with $X = I \cup N(I)$. Lemma 2.4 implies that

$$2\alpha(G) = |\text{diadem}(G)| + |\text{nucleus}(G)| \leq |\text{diadem}(G[X])| + |\text{nucleus}(G[X])|.$$

Theorem 1.2 (ii) gives that $G[X]$ is König-Egerváry, so by Corollary 1.5 we have $|\text{diadem}(G[X])| + |\text{nucleus}(G[X])| = 2\alpha(G[X])$ implying that $\alpha(G) \leq \alpha(G[X])$. It follows by Theorem 1.2 (i) we must have $\alpha(G) = \alpha(G[X])$, so again we know that $\alpha(G[X^c]) = 0$ which finishes this part of the proof.

The implications $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$ are given in Theorem 1.4 and in Theorem 1.5. \square

4 A bound on $\alpha(G)$

Proof (of Theorem 1.7). Let I be a maximum critical independent set in G and $X = I \cup N(I)$. By Theorem 1.2 (ii), $G[X]$ is a König-Egerváry graph

so by Theorem 1.5 we have

$$|\text{nucleus}(G[X])| + |\text{diadem}(G[X])| = 2\alpha(G[X]) \leq 2\alpha(G).$$

Now by Lemma 2.4 we must have

$$|\text{nucleus}(G)| + |\text{diadem}(G)| \leq |\text{nucleus}(G[X])| + |\text{diadem}(G[X])|$$

and the theorem follows. \square

Combining Theorem 1.7 and the inequality $2\alpha(G) \leq |\text{core}(G)| + |\text{corona}(G)|$ proven in [4], the following corollary is immediate.

Corollary 4.1. *For any graph G ,*

$$|\text{nucleus}(G)| + |\text{diadem}(G)| \leq 2\alpha(G) \leq |\text{core}(G)| + |\text{corona}(G)|.$$

These upper and lower bounds are quite interesting. The fact that every critical independent set is contained in a maximum independent set implies that $\text{diadem}(G) \subseteq \text{corona}(G)$ for all graphs G . However, the graph G_2 in Figure 2 has $\text{core}(G_2) \subsetneq \text{nucleus}(G_2)$ while the graph G in Figure 1 has $\text{nucleus}(G) = \{a, b, c\} \subsetneq \text{core}(G) = \{a, b, c, h\}$.

5 Acknowledgements

Many thanks to my advisor László Székely for feedback on initial versions of this manuscript. Partial support from the NSF DMS under contract 1300547 is gratefully acknowledged.

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